Columbia Business School
Briger Family Digital Finance Lab

Trading with Concave (Cross-)Impact

Natascha Hey
Columbia Business School

This presentation was supported by Capital Fund Management

FutureAlpha 2026

Outline

Trading in a competitive world

1. How much does it cost to trade? **Price impact** and other costs.
2. Why do we trade? Price prediction and **alpha signals**.
3. Alpha is competitive, assets under management are limited.

From empirical findings to market insights

1. *Trading with Concave Price Impact and Impact Decay*. (H., Mastromatteo, Muhle-Karbe, Webster)
2. *Concave Cross-Impact*. (H., Mastromatteo, Muhle-Karbe)

An optimal trading strategy should ...

Outline

Trading in a competitive world

1. How much does it cost to trade? **Price impact** and other costs.
2. Why do we trade? Price prediction and **alpha signals**.
3. Alpha is competitive, assets under management are limited.

From empirical findings to market insights

1. *Trading with Concave Price Impact and Impact Decay*. (H., Mastromatteo, Muhle-Karbe, Webster)
2. *Concave Cross-Impact*. (H., Mastromatteo, Muhle-Karbe)

An optimal trading strategy should ...

Outline

Trading in a competitive world

1. How much does it cost to trade? **Price impact** and other costs.
2. Why do we trade? Price prediction and **alpha signals**.
3. Alpha is competitive, assets under management are limited.

From empirical findings to market insights

1. *Trading with Concave Price Impact and Impact Decay*. (H., Mastromatteo, Muhle-Karbe, Webster)
2. *Concave Cross-Impact*. (H., Mastromatteo, Muhle-Karbe)

An optimal trading strategy should ...

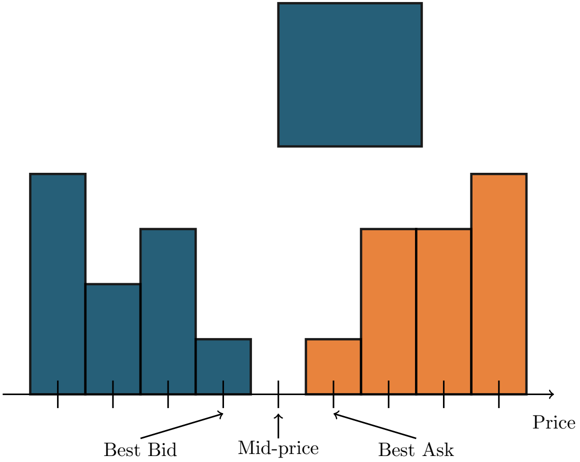
Transaction Costs

- **Explicit transaction costs** are easy to measure and usually known in advance. They include commissions to broker, taxes, exchange fees, etc.

- **Implicit transaction costs** are more subtle and require statistical estimation. For large orders costs through *bid-ask spread* are usually negligible compared to *price impact*.

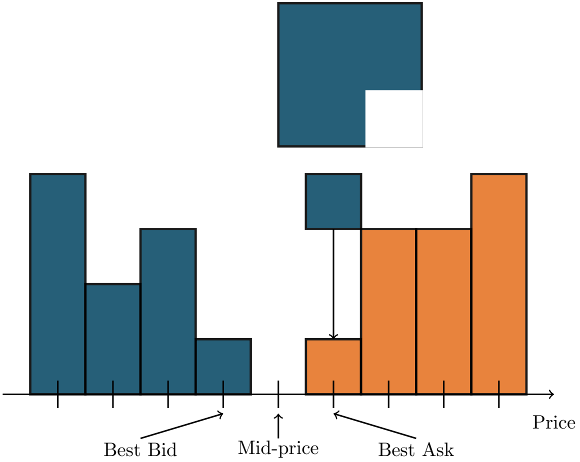
What is Price Impact?

Large institutional investors want to buy a large quantity.

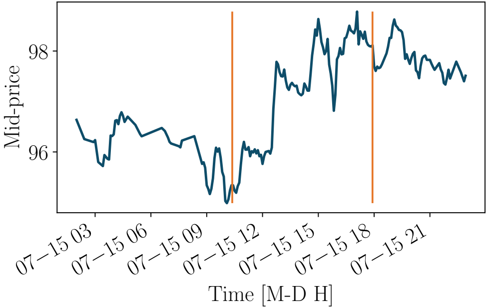


What is Price Impact?

Large institutional investors split their meta-orders into child-orders.



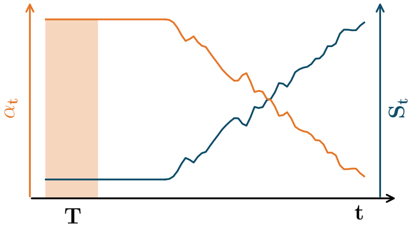
Example - Price Movement Caused by Trading



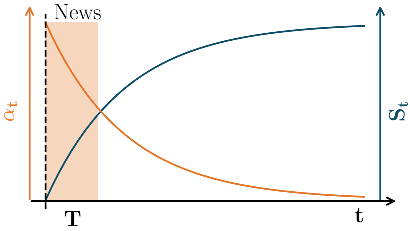
The price forecast is commonly called an alpha signal.

$$\boxed{\alpha_t = \mathbb{E}_t\left[S_{T'} - S_t\right]}$$

for some prediction horizon T' larger than the execution horizon T, where S_t is the fundamental price at time t.



(a) Constant: Long-term traders do not assume intra-day decay.



(b) Decaying: News signals may decay during execution.

Large institutional investors cannot grow indefinetly.

- Alpha generation is competitive.
- Traded volume is limited by price impact.

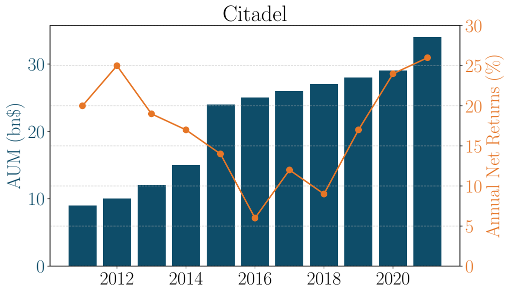


Figure: Financial Times, “How Ken Griffin rebuilt Citadel’s ramparts”, 2021.

"There are more hedge funds than Burger Kings."

— Financial Times, 2023.[1]

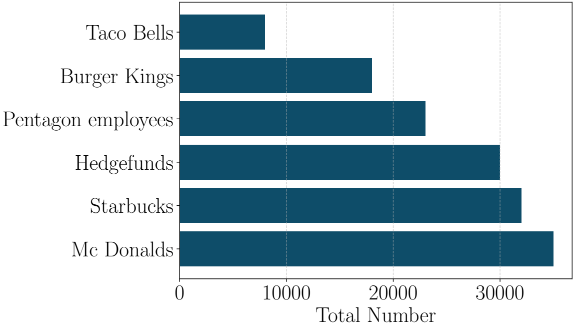


Figure: Number of hedge funds in the USA 2023.[1]

Motivation

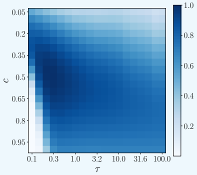
Portfolio managers optimize their returns with respect to transaction costs.

- Alpha signals predict return regardless of trading.
- (Implicit) Transaction costs are caused by trading.

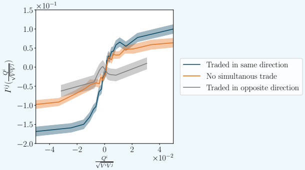
Goal: Maximize expected profit by adapting trading strategy.

Trading in a competitive world

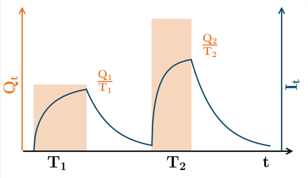
Empirical Estimation



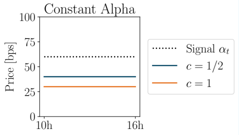
Concave Cross-Impact



A Tractable Price Impact Model



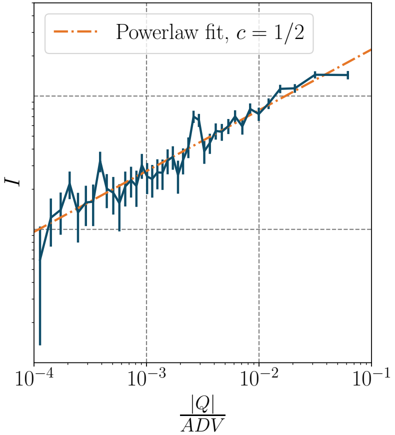
Transaction Cost Analysis



Price impact conditioned on traded volume

$$I^{\max} = \mathbb{E}\left[\operatorname{sign}(Q)\cdot(P_{\text{end}} - P_{\text{start}})\,\middle|\,\frac{|Q|}{ADV}\right]$$
$$\propto \lambda\left(\frac{|Q|}{ADV}\right)^{c}$$

- c - Concavity parameter [1]
- P - Mid-price
- Q - Meta-order volume
- ADV - Average daily traded volume
- λ - Push factor/ Kyle's Lambda



[1] Almgren et al. (2005), Toth et al. (2011), Sato & Kanazawa (2024)

Price impact is a nonlinear function of all order flow[1]

$$I_t = h(J_t), \quad h(x) = \mathrm{sign}(x)|x|^c \qquad (1)$$

and the exponentially weighted moving average J_t

$$dJ_t = -\frac{1}{\tau} J_t dt + \lambda_t dQ_t, \quad J_0 = 0. \qquad (2)$$

- dQ_t - Traded volume at time t
- τ - Impact decay time

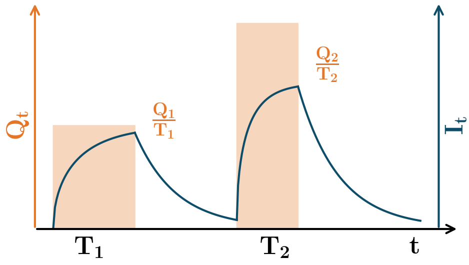
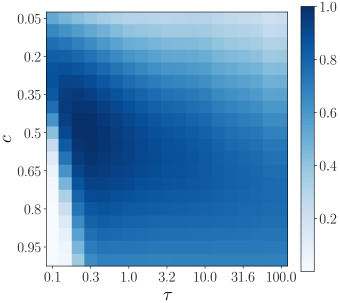


Figure: Sample impact trajectory with uniform meta-order trades.

[1] For $c = 1$: Obizhaeva , Wang (2013). For $c \leq 1$ Alfonsi, Fruth, Schied (2010)

- **Proprietary CFM dataset on futures**
- $1.9 \cdot 10^5$ Meta-Orders between 2012-2022 with ordersize $[0.01\% - 10\%]$ of ADV
- Example of the data

Product	Date	T^{start}	T^{end}	P^{start}	P^{end}	σ	Q	V	N
BRENT 0	2022-11-19	11:00	15:30	80.2	80.6	1.15	90	$1.7{\cdot}10^5$	51
BRENT 1	2022-12-10	10:00	14:00	84.9	84.3	0.8	200	$1.6{\cdot}10^5$	63
BRENT 2	2022-12-10	14:00	20:30	84.8	84.6	0.9	25	$6.5{\cdot}10^4$	39



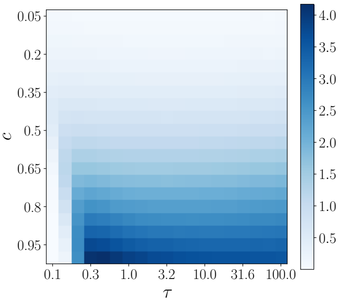


Figure: Statistical **significance ratio** $\frac{R^2(c,\tau)}{R^2(\hat{c},\hat{\tau})}$;
Point estimates: $\hat{c} = 0.5$ and $\hat{\tau} = 0.3$.

Figure: Model's prefactor $\lambda(c,\tau)$;
At the point estimates $\lambda(\hat{c},\hat{\tau}) \approx 1.5$.

How much alpha can we pay to balance optimally impact and alpha?

Strategy	α (bps)	I (bps)
A (Macro)	60	?
B (News)	90	?

Maximizing Portfolio's Profit and Loss

A **risk-neutral** statistical arbitrage portfolio maximizes w.r.t. executed volume

$$\boxed{\mathbb{E}\left[Y_T\right] = \mathbb{E}\left[\int_0^T (\alpha_t - I_t) dQ_t\right].} \tag{3}$$

Goal: Maximize expected $P\&L$ by adjusting the trade size over the execution time T

$$\sup_Q \mathbb{E}\left[\int_0^T (\alpha_t - I_t) dQ_t\right].$$

Maximize expected $P\&L$ by adjusting the trade size over the execution time T

$$\sup_{Q} \mathbb{E}\left[\int_0^T (\alpha_t - I_t)dQ_t\right].$$

Trick: Pointwise maximization by mapping into "impact space"[1]

$$dQ_t = \lambda_t^{-1}\tau^{-1}J_t dt + \lambda_t^{-1}dJ_t, \quad I_t = h(J_t), \quad \lambda_t = e^{\gamma_t} \tag{4}$$

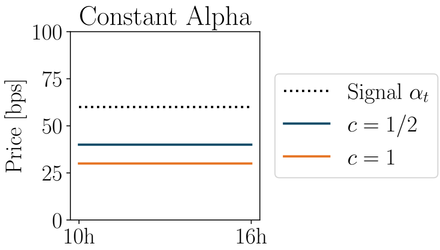
where γ_t is a liquidity parameter. The optimal impact is

$$\boxed{I_t^* = \frac{\tau^{-1} + \gamma_t'}{(1+c)\tau^{-1} + \gamma_t'}\alpha_t - \frac{1}{(1+c)\tau^{-1} + \gamma_t'}\alpha_t'.} \tag{5}$$

[1] Fruth et al. (2013) and Gueant, Lehalle (2013)

TCA - Long-Term strategy

Strategy	α (bps)	I^* (bps)
A (Macro)	60	40
B (News)	90	72



- Strategy A constant alpha $\alpha_t = \alpha$

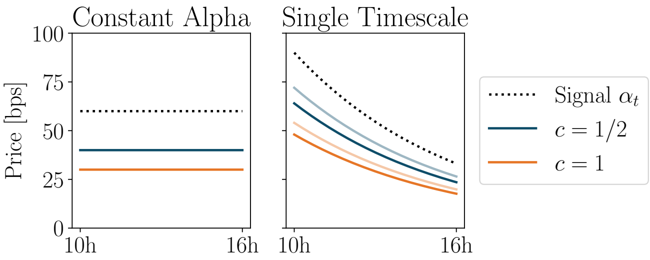
$$I^* = \frac{1}{1+c}\alpha = \frac{2}{3}\alpha$$

Strategy	α (bps)	τ/θ	I^* (bps)
A (Macro)	60	0	40
B (News)	90	1/5	72

- Strategy B with decaying alpha

$$d\alpha_t = -\frac{1}{\theta}\alpha_t dt + \sigma dW, \quad \alpha = \alpha_0$$

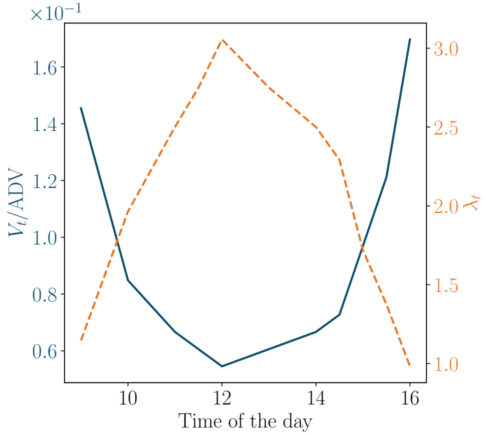
$$I_t^* = \frac{2}{3}\alpha_t\left(1 + \frac{\tau}{\theta}\right), \quad I = I_0$$



Traded volume and volatility...

- ... increase at **Open** due to price discovery, overnight orders etc..
- ... decrease during lunch.
- ... increase at **Close** due to portfolio rebalancing, trade settlement, valuation at close price, etc..

$\rightarrow$ **The push-factor λ_t is a function of liquidity and volatility.**

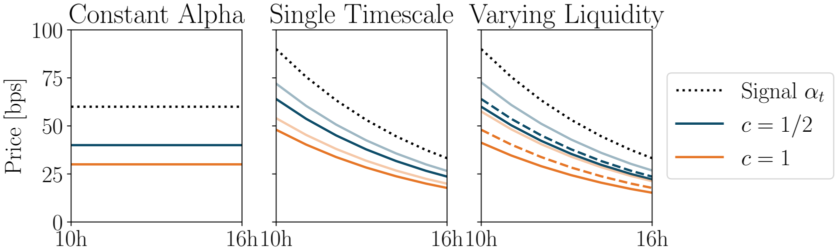


Strategy	α (bps)	τ/θ	I^* (bps)
Constant Liquidity	90	1/15	64
Dropping Liquidity	90	1/15	72
Raising Liquidity	90	1/15	60

- Strategy B with varying liquidity

$$I_t^* = \frac{\alpha_t}{(3/2) + \gamma_t' \tau} \left(1 + \tau \gamma_t' + \frac{\tau}{\theta}\right) \quad (6)$$

where $\lambda_t = e^{\gamma_t}$



No Price Manipulation Condition

Time-dependent liquidity can potentially lead to price manipulation. In periods of low liquidity, a trader could buy the asset and push market prices up and unwind their position in periods of higher liquidity without depressing market prices. [2]

An arbitrage-free model requires:

- The integrand in the P&L to be strictly concave.
- Price impact cannot decay faster than liquidity changes.

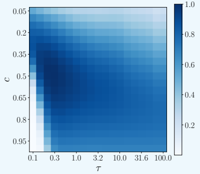
$$(1+c) > |\tau\gamma_t'| \tag{7}$$

[2] Fruth et al. (2013)

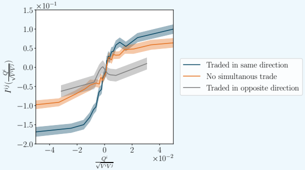
Outline

Trading in a competitive world

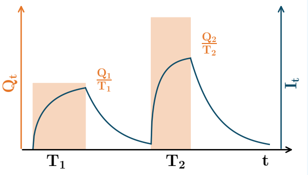
Empirical Estimation



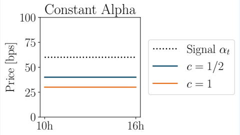
Concave Cross-Impact



A Tractable Price Impact Model



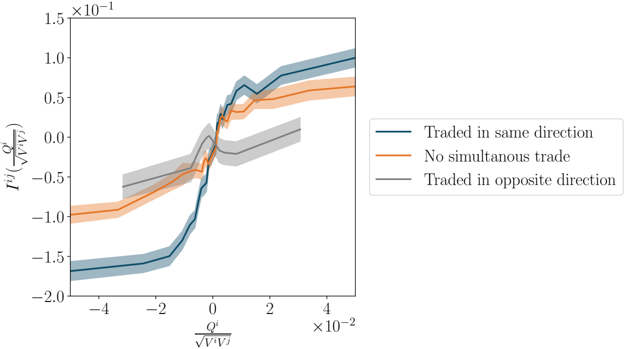
Transaction Cost Analysis



Cross-Impact - Empirical Evidence

Cross-Impact[1] of asset i on asset j with return correlation > 90%

$$I_{ij} = \mathbb{E}\left[\epsilon_i \left(P_j^{end} - P_j^{start}\right) \middle| \frac{Q_i}{\sqrt{V_i V_j}}\right] \tag{8}$$



[2] Pasquariello & Vega (2013), Mastromatteo et al., Wang & Guhr (2017), Tomas et al. (2022), Le Coz et al. (2024)

Price Impact $\boldsymbol{I}_t = (I_t^1, \ldots, I_t^d)^\top$ in each asset $i = 1, \ldots, d$ is

$$\begin{aligned} \boldsymbol{I}_t = \boldsymbol{L} h(\boldsymbol{J}_t), \quad & \text{where } h(\boldsymbol{J}_t) = (h(J_t^1), \ldots, h(J_t^d))^\top \in \mathbb{R}^d \\ & \text{and} \quad h(x) = \operatorname{sign}(x)|x|^c. \end{aligned} \tag{9}$$

The multivariate moving average $\boldsymbol{J}_t = (J_t^1, \ldots, J_t^d)^\top$ is:

$$d\boldsymbol{J}_t = -\boldsymbol{\tau}^{-1} \boldsymbol{J}_t dt + \boldsymbol{\Lambda} d\boldsymbol{Q}_t, \quad \boldsymbol{\tau}, \boldsymbol{\Lambda} \in \mathbb{R}^{d \times d}. \tag{10}$$

- $\boldsymbol{Q}_t = (Q_t^1, \ldots, Q_t^d)^\top$: Trader's holdings.
- $\boldsymbol{L} = (\boldsymbol{L}^1, \ldots, \boldsymbol{L}^d) \in \mathbb{R}^{d \times d}$: Liquidity factors.
- $\boldsymbol{\Lambda}$: Matrix of correlation structure and push-factors.
- $\boldsymbol{\tau}$: Matrix of decay time-scales.

The General Model is Not Tractable

Passage of the risk-neutral PnL into "impact space"

$$\sup_{(\boldsymbol{Q}_t)_{t\in[0,T]}} \mathbb{E}\left[\int_0^T (\boldsymbol{\alpha}_t - \boldsymbol{I}_t)^\top d\boldsymbol{Q}_t\right] = \sup_{(\boldsymbol{J}_t)_{t\in[0,T]}} \mathbb{E}\Big[\int_0^T \left(\bar{\boldsymbol{\alpha}}_t^\top \boldsymbol{J}_t - h(\boldsymbol{J}_t)^\top \boldsymbol{\zeta} \boldsymbol{J}_t\right) dt - \int_0^T h(\boldsymbol{J}_t)^\top \boldsymbol{\theta} d\boldsymbol{J}_t + \boldsymbol{\alpha}_T^\top \boldsymbol{\Lambda}^{-1} \boldsymbol{J}_T\Big]. \tag{11}$$

Define rotations to latent space:

$$\boldsymbol{\zeta} = \boldsymbol{L}^\top (\boldsymbol{\tau}\boldsymbol{\Lambda})^{-1}, \qquad \boldsymbol{\theta} = \boldsymbol{L}^\top \boldsymbol{\Lambda}^{-1}, \quad \text{and} \quad \bar{\boldsymbol{\alpha}}_t = \boldsymbol{\zeta}^\top \boldsymbol{L}^{-1} \boldsymbol{\alpha}_t - \boldsymbol{\theta}^\top \boldsymbol{L}^{-1} \boldsymbol{\alpha}'_t, \tag{12}$$

- The first term can be pointwise maximized
- Problem: $h(J_t^i) dJ_t^k$ have no common antiderivative.

What are the conditions on θ and ζ to have a well-posed problem?

The General Model is Not Tractable

Passage of the risk-neutral PnL into "impact space"

$$\sup_{(\boldsymbol{Q}_t)_{t\in[0,T]}} \mathbb{E}\left[\int_0^T (\boldsymbol{\alpha}_t - \boldsymbol{I}_t)^\top d\boldsymbol{Q}_t\right] = \sup_{(\boldsymbol{J}_t)_{t\in[0,T]}} \mathbb{E}\Big[\int_0^T \Big(\bar{\boldsymbol{\alpha}}_t^\top \boldsymbol{J}_t - h(\boldsymbol{J}_t)^\top \boldsymbol{\zeta} \boldsymbol{J}_t\Big)\, dt - \int_0^T h(\boldsymbol{J}_t)^\top \boldsymbol{\theta} d\boldsymbol{J}_t + \boldsymbol{\alpha}_T^\top \boldsymbol{\Lambda}^{-1} \boldsymbol{J}_T\Big]. \tag{13}$$

Define rotations to latent space:

$$\boldsymbol{\zeta} = \boldsymbol{L}^\top (\boldsymbol{\tau}\boldsymbol{\Lambda})^{-1}, \qquad \boldsymbol{\theta} = \boldsymbol{L}^\top \boldsymbol{\Lambda}^{-1}, \quad \text{and} \quad \bar{\boldsymbol{\alpha}}_t = \boldsymbol{\zeta}^\top \boldsymbol{L}^{-1} \boldsymbol{\alpha}_t - \boldsymbol{\theta}^\top \boldsymbol{L}^{-1} \boldsymbol{\alpha}_t', \tag{14}$$

- The first term can be pointwise maximized
- Problem: $h(J_t^i) dJ_t^k$ have no common antiderivative.

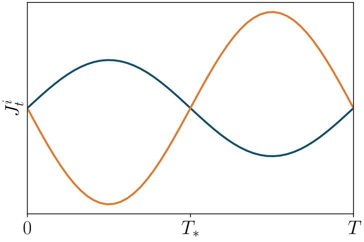
What are the conditions on θ and ζ to have a well-posed problem?

No Price Manipulation Conditions[1]

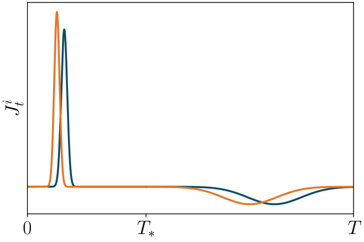
Inspired by Gatheral '10 define trading costs of a round-trip trade by

$$C_T = \int_0^T h(\boldsymbol{J}_t)^\top \boldsymbol{\zeta} \boldsymbol{J}_t dt + \int_0^T h(\boldsymbol{J}_t)^\top \boldsymbol{\theta} d\boldsymbol{J}_t. \tag{15}$$

Derive "smart strategies" that isolate the $\boldsymbol{\theta}$ and ζ terms to rule out negative costs.



(a) The Symmetric Strategy for ζ



(b) The Impulsive Strategy for θ

[2]Huberman & Stanzl (2004), Fruth et al. (2013)

Conditions Reduce Space of Models

When considering **cross-impact** and **no-arbitrage** conditions then the space of models that can be solved analytically reduces to the following cases:

Cases	d	θ	ζ
Decoupled	$\mathbb{N}^+$	$\mathrm{diag}(\theta_{11}, ..., \theta_{dd})$	$\mathrm{diag}(\zeta_{11}, ..., \zeta_{dd})$
Bivariate	2	$\mathrm{diag}(\theta_{11}, ..., \theta_{dd})$	$(\zeta_{ij})_{i \neq j} < \zeta_{ii}$
General	$\mathbb{N}^+$	$\mathrm{diag}(\theta_{11}, ..., \theta_{dd})$	$\mathrm{diag}(\zeta_{11} - \epsilon, ..., \zeta_{dd} - \epsilon) + \epsilon$

Trading Strategy for Correlates Assets in the Decoupled Case

The optimal **impact state** and **trading rate** for $t \in (0, T)$ is

$$\boldsymbol{I}_t^* = \frac{\boldsymbol{\alpha}_t}{1+c}, \quad \frac{d\boldsymbol{Q}_t^*}{dt} = (\mathbf{L}^\top)^{-1} \zeta h^{-1} \left(\frac{\boldsymbol{L}^{-1} \boldsymbol{\alpha}_t}{1+c} \right) \tag{16}$$

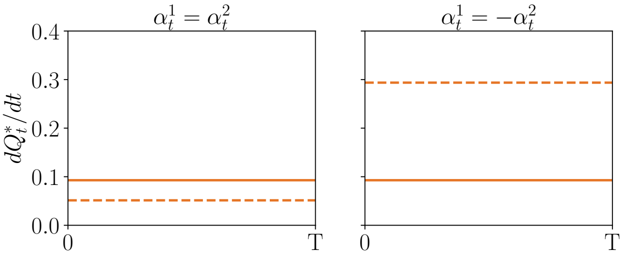
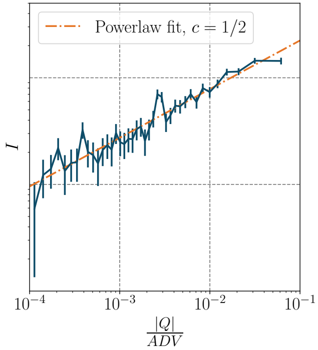


Figure: **Optimal Trading rate** for (anti-) aligned alpha signals and return correlation of 0.6.
Solid line: Without Cross-Impact, Dashed line: With Cross-Impact

Summary

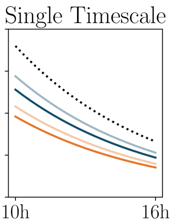
An optimal trading strategy should

- reflect the concave nature of market impact;
- balance alpha decay and impact decay across the full execution horizon;
- adapt to changing market liquidity;
- remain arbitrage-free by construction;
- account for cross-asset effects and optimize execution at the portfolio level;



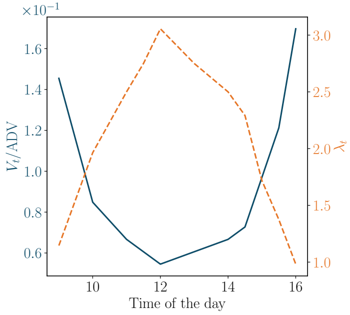
An optimal trading strategy should

- reflect the concave nature of market impact;
- balance alpha decay and impact decay across the full execution horizon;
- adapt to changing market liquidity;
- remain arbitrage-free by construction;
- account for cross-asset effects and optimize execution at the portfolio level;



An optimal trading strategy should

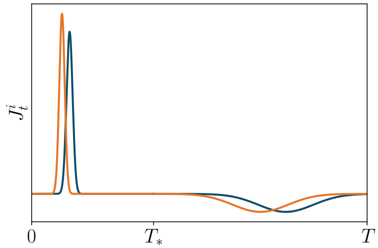
- reflect the concave nature of market impact;
- balance alpha decay and impact decay across the full execution horizon;
- adapt to changing market liquidity;
- remain arbitrage-free by construction;
- account for cross-asset effects and optimize execution at the portfolio level;



Summary

An optimal trading strategy should

- reflect the concave nature of market impact;
- balance alpha decay and impact decay across the full execution horizon;
- adapt to changing market liquidity;
- remain arbitrage-free by construction;
- account for cross-asset effects and optimize execution at the portfolio level;



An optimal trading strategy should

- reflect the concave nature of market impact;
- balance alpha decay and impact decay across the full execution horizon;
- adapt to changing market liquidity;
- remain arbitrage-free by construction;
- account for cross-asset effects and optimize execution at the portfolio level;