Autoregressive conditional betas

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Future Alpha, New York
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Models with time-varying betas

$$
\begin{aligned}
y_t &= \beta_{1,t} x_{1,t} + \cdots + \beta_{p,t} x_{p,t} + v_t \\
v_t &= g_t \eta_t, g_t^2 = V(y_t | \Omega_{t-1})
\end{aligned}
$$

Simple but useful in finance.

Tracking portfolios by Blasques, Francq and Laurent (JoE, 2024)

Six factors model:

$$\begin{aligned} y_t =& \beta_{CST,t} + \beta_{MKT,t} MKT_t + \beta_{SMB,t} SMB_t + \beta_{HML,t} HML_t \\ &+ \beta_{RMW,t} RMW_t + \beta_{CMA,t} CMA_t + \beta_{MOM,t} MOM_t + v_t \end{aligned}$$

$$\begin{aligned} Z_{t+1|t} =& \beta_{MKT,t+1|t} MKT_{t+1} + \beta_{SMB,t+1|t} SMB_{t+1} + \beta_{HML,t+1|t} HML_{t+1} \\ &+ \beta_{RMW,t+1|t} RMW_{t+1} + \beta_{CMA,t+1|t} CMA_{t+1} + \beta_{MOM,t+1|t} MOM_{t+1} \end{aligned}$$

We seek for the model with the smallest sample mean squared tracking error (MSE) $TE_{k,t+1} = \frac{1}{n} \sum_{t=1}^{n} (y_{t+1} - Z_{t+1|t})^2$.

Precision matrix $\Sigma_{t-1|t}^{-1} = B_t' G_t^{-1} B_t$ by Darolles, Francq and Laurent (2018, JoE)

Step 1: $y_{1,t} = \beta_{10} + g_{11,t}\eta_{1,t}$

Step 2: $y_{2,t} = \beta_{20} + \beta_{21,t}y_{1,t} + g_{22,t}\eta_{2,t}$

Step 3: $y_{3,t} = \beta_{30} + \beta_{31,t}y_{1,t} + \beta_{32,t}y_{2,t} + g_{33,t}\eta_{3,t}$

$$B_t = \begin{bmatrix} 1 & 0 & 0 \\ -\beta_{21,t} & 1 & 0 \\ -\beta_{31,t} & -\beta_{32,t} & 1 \end{bmatrix} \quad G_t^{-1} = \begin{bmatrix} \frac{1}{g_{11,t}} & 0 & 0 \\ 0 & \frac{1}{g_{22,t}} & 0 \\ 0 & 0 & \frac{1}{g_{33,t}} \end{bmatrix}.$$

$$\Sigma_t^{-1} = \begin{bmatrix} \frac{1}{g_{11,t}} + \frac{\beta_{21,t}^2}{g_{22,t}} + \frac{\beta_{31,t}^2}{g_{33,t}} & -\frac{\beta_{21,t}}{g_{22,t}} & -\frac{\beta_{31,t}}{g_{33,t}} \\ -\frac{\beta_{21,t}}{g_{22,t}} & \frac{1}{g_{22,t}} + \frac{\beta_{32,t}^2}{g_{33,t}} & -\frac{\beta_{32,t}}{g_{33,t}} \\ -\frac{\beta_{31,t}}{g_{33,t}} & -\frac{\beta_{32,t}}{g_{33,t}} & \frac{1}{g_{33,t}} \end{bmatrix}.$$

Time-varying betas: $y_t = \beta_{1,t}x_{1,t} + \cdots + \beta_{p,t}x_{p,t} + v_t$

Example 1 (Rolling windows):

$$y_t = \beta_1 x_{1,t} + \cdots + \beta_p x_{p,t} + v_t$$

Example 2 (Interaction variables):

$$\beta_{i,t} = \varpi_i + \gamma_{1,i} z_{1,t-1} + \cdots + \gamma_{q,i} z_{q,t-1}.$$

Example 3 (State Space - RW Model):

$$\beta_{i,t} = \beta_{i,t-1} + \delta_i u_{i,t},$$

where $\delta_i \geq 0$, $u_{i,t} \overset{i.i.d.}{\sim} N(0,1)\ \forall i$ and $u_{i,t} \perp u_{j,t}\ \forall i \neq j$.

Time-varying betas: $y_t = \beta_{1,t}x_{1,t} + \cdots + \beta_{p,t}x_{p,t} + v_t$

Example 4 (Dynamic conditional betas):

Engle (2016) shows that (y_t, x_t), $t = 1, \ldots, T$, is a vector of $1 + p$ random variables, assumed to be conditionally Gaussian

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} |\mathcal{F}_{t-1} \sim N\left(\begin{pmatrix} \mu_{y_t} \\ \mu_{x_t} \end{pmatrix}, \begin{pmatrix} \Sigma_{yy,t} & \Sigma_{yx,t} \\ \Sigma_{xy,t} & \Sigma_{xx,t} \end{pmatrix}\right).$$

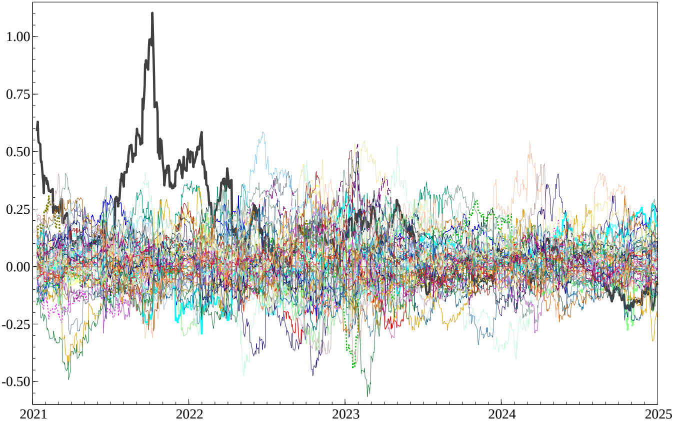
Thus $y_t|x_t \sim N\left(\mu_{y_t} + \boldsymbol{\Sigma}_{yx,t}\boldsymbol{\Sigma}_{xx,t}^{-1}(x_t - \mu_{x_t}), \Sigma_{yy,t} - \Sigma_{yx,t}\Sigma_{xx,t}^{-1}\Sigma_{xy,t}\right)$

$\Rightarrow \boldsymbol{\beta}_t = \boldsymbol{\Sigma}_{xx,t}^{-1}\boldsymbol{\Sigma}_{xy,t}$ can be obtained by first estimating a DCC GARCH model on (y_t, x_t) - **NOT** a natural way to specify the parameter dynamics !

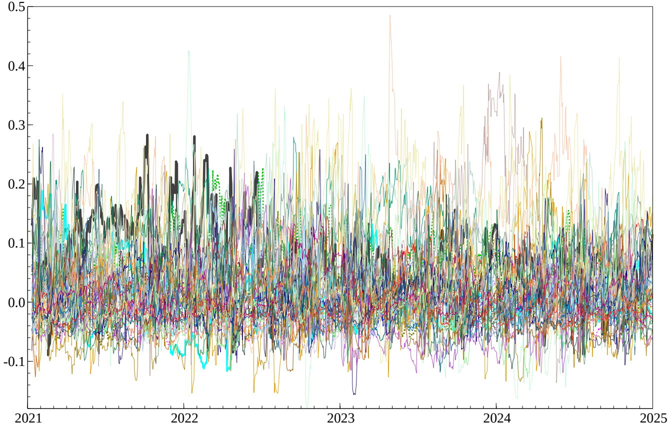
Minimum Variance Portfolio

- $\omega^*_{t+1} = (\omega^*_{t+1,1}, \ldots, \omega^*_{t+1,n})$
- GMVP: $\omega^*_{t+1} = \frac{\Sigma^{-1}_{t+1|t}\iota}{\iota'\Sigma^{-1}_{t+1|t}\iota}, \iota'\omega^*_{t+1} = 1$
- Daily returns on 50 US stocks from 2005-02-28 to 2024-12-31, i.e. 4,991 observations.
- Last 1,000 observations as test period.

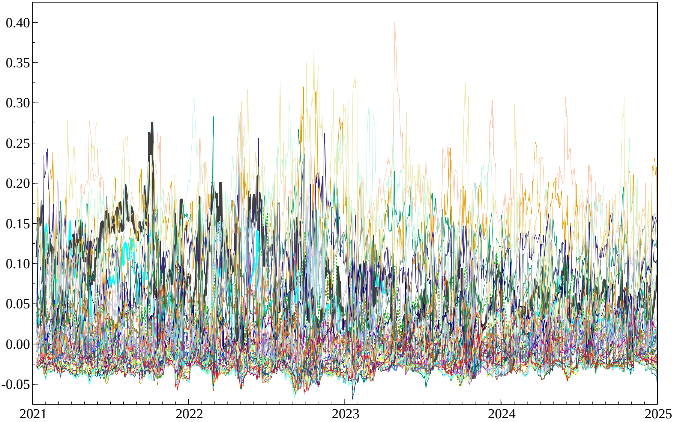
Weights GMVP - Empirical Cov on 100 days



Weights GMVP - DCC



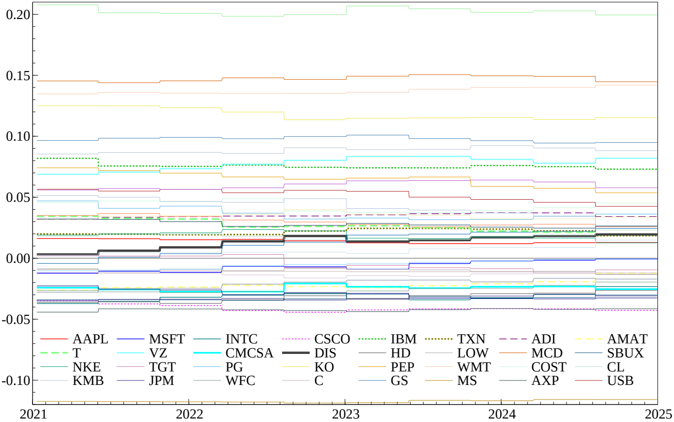
Weights GMVP - DECO



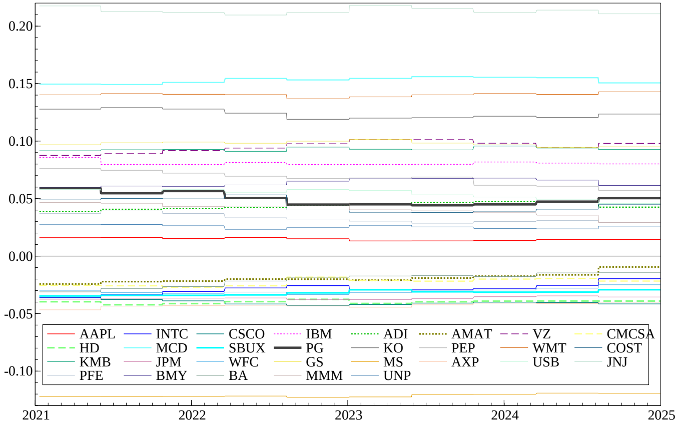
Minimum Variance Portfolio

- Dynamic version of Kempf and Memmel (2006):
  $y_t = X_t' \beta_t + \epsilon_t$,
- where $y_t = r_{n,t}$, $X_t = (1, r_{n,t} - r_{1,t}, \ldots, r_{n,t} - r_{n-1,t})'$.
- $\omega^*_{t+1,i} = \beta_{t+1|t,i} \; \forall i = 1, \ldots, n-1$ and
  $\omega^*_{t+1,n} = 1 - \sum_{j=1}^{n-1} \beta_{t+1|t,j}$.

Weights GMVP - Lasso (CV, 5 folds)



Weights GMVP - Autometrics at 1%



Models with the smallest sample mean squared tracking error

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Ox 10.04 (macOS_64/Parallel) (C) J.A. Doornik, 1994-2025 (oxlang.dev)
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MulCom package version 3.01, object created on 22-02-2026

  ------  MODEL CONFIDENCE SET ESTIMATION  ------
Number of models:      l=5
Sample size:           n=1000
Loss function:         ident
Test Statistic:        Range
Resample by:           BlockBootResamp
Bootstrap parameters: B=10000 (resamples), d=5 (block length)

Model Name                     ident(*10^3)   MCS p-val.
Lasso-KM                           0.61429    1.0000 *
Auto-KM                            0.62263    0.1898 **
DCC                                0.65128    0.1898 **
--------------------------------------------------------
DECO                               0.91946    0.0000
COV_100                            0.93566    0.0000
```

Models with the smallest sample mean squared tracking error

Method	# Assets	# TV weights	Var $\times 10^3$	p-value	TO_{ann}
Lasso	32	0	0.61429	1.0000	0.0910
Auto	29	0	0.62263	0.1898	0.0602
DCC	50	50	0.65128	0.1898	76.1992
COV100	50	50	0.91946	0.0000	86.6967
DECO	50	50	0.93566	0.0000	45.2824

Note: The number of assets receiving a non-zero portfolio weight for each strategy is reported in the second column (# Assets). The third column (# TV weights) reports the number of time-varying weights. The fourth column reports the empirical variance of the portfolios while the last column is the p-value of the MCS test.

The ACB model

Blasques, Francq and Laurent (JoE, 2024) (BFL) propose the Autoregressive Conditional Beta (ACB) model

$$\begin{aligned}
y_t &= \beta_{1,t}x_{1,t} + \cdots + \beta_{p,t}x_{p,t} + v_t, \quad v_t = g_t\eta_t,\\
g_{t+1}^2 &= \omega_0 + \alpha_0 v_t^2 + \beta_0 g_t^2,\\
\beta_{i,t+1} &= \varpi_{0i} + \xi_{0i}\frac{v_t x_{i,t}}{\mu_{0i,t}^2 + g_{i,t}^2} + c_{0i}\beta_{i,t} + \gamma_{01,i}z_{1,t} + \ldots + \gamma_{0q,i}z_{q,t},
\end{aligned}$$

with regressors $\boldsymbol{x}_t = (x_{1,t}, \ldots, x_{p,t})^\top$ and exogenous variables $\boldsymbol{z}_t = (z_{1,t}, \ldots, z_{q,t})^\top$.

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\end{aligned}
$$

with regressors $\boldsymbol{x}_t = (x_{1,t}, \ldots, x_{p,t})^\top$ and exogenous variables $\boldsymbol{z}_t = (z_{1,t}, \ldots, z_{q,t})^\top$.

Score Driven models

Assume that y_t follows a conditional density $p(y_t|f_t, \mathcal{F}_{t-1}, \theta)$, where f_t is a time-varying parameter of interest.

$$f_{t+1} = \varpi + \xi S(f_t) \frac{\partial \log p(y_t|f_t, \mathcal{F}_{t-1}, \theta)}{\partial f_t} + c f_t,$$

where ϖ, ξ and c are unknown parameters and $S(f_t)$ is the inverse of the conditional information matrix.

$\rightarrow$ The scaled score is the updating mechanism in this approach.

Applying the SD approach for the beta parameters

Let us define a SD model for β_{it} in the regression model

$$y_t = \beta_{1t}x_{1t} + \cdots + \beta_{pt}x_{pt} + v_t,$$

where $v_t = g_t\eta_t$, $g_t^2 = E_{t-1}v_t^2$, η_t iid $\mathcal{N}(0,1)$. Let $E_{t-1}x_{it} = \mu_{it}$ and $E_{t-1}x_{it}^2 = g_{it}^2 + \mu_{it}^2$.

We have $l_t := \log p(y_t|\mathcal{F}_{t-1},\theta) \approx -\frac{1}{2}\left\{\frac{v_t^2}{g_t^2} + \log g_t^2\right\}$,

$$\frac{\partial l_t}{\partial \beta_{it}} = \frac{v_t x_{it}}{g_t^2}, \quad S(\beta_{it}) = -\left(E_{t-1}\frac{\partial^2 l_t}{\partial^2 \beta_{it}}\right)^{-1} = \frac{g_t^2}{\mu_{it}^2 + g_{it}^2}.$$

Therefore the updating mechanism $S(\beta_{it})\frac{\partial l_t}{\partial \beta_{it}} = \frac{v_t x_{it}}{\mu_{it}^2 + g_{it}^2}$.

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$$y_t = \beta_{1t}x_{1t} + \cdots + \beta_{pt}x_{pt} + v_t,$$

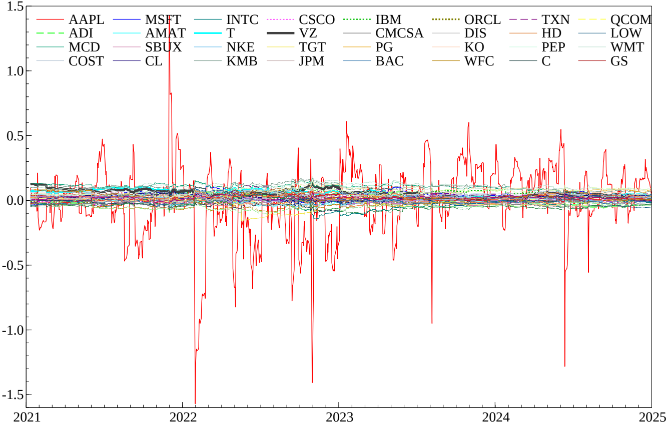
where $v_t = g_t\eta_t$, $g_t^2 = E_{t-1}v_t^2$, η_t iid $\mathcal{N}(0,1)$. Let $E_{t-1}x_{it} = \mu_{it}$ and $E_{t-1}x_{it}^2 = g_{it}^2 + \mu_{it}^2$.

We have $l_t := \log p(y_t|\mathcal{F}_{t-1},\theta) \approx -\frac{1}{2}\left\{\frac{v_t^2}{g_t^2} + \log g_t^2\right\}$,

$$\frac{\partial l_t}{\partial \beta_{it}} = \frac{v_t x_{it}}{g_t^2}, \quad S(\beta_{it}) = -\left(E_{t-1}\frac{\partial^2 l_t}{\partial^2 \beta_{it}}\right)^{-1} = \frac{g_t^2}{\mu_{it}^2 + g_{it}^2}.$$

Therefore the updating mechanism $S(\beta_{it})\frac{\partial l_t}{\partial \beta_{it}} = \frac{v_t x_{it}}{\mu_{it}^2 + g_{it}^2}$.

Weights GMVP - ACB



Unidentifiability of the constant betas

$$\beta_{i,t+1} = \varpi_{0,i} + \xi_{0,i} \frac{v_t x_{i,t}}{\mu_{0,i}^2 + g_{i,t}^2} + c_{0,i}\beta_{i,t}$$

$\beta_{i,t}$ is constant if and only if

$$\xi_{0i} = 0.$$

Note that when this relation holds the parameter c_{0i} is not well defined because the model remains the same for all values of (ϖ_{i0}, c_{0i}) such that $\varpi_{i0}/(1 - c_{0i})$ is fixed:

$$\beta_i = \varpi_{0i} + 0 \cdot \frac{v_t x_{i,t}}{\mu_{0i}^2 + g_{i,t}^2} + c_{0i}\beta_i$$

Selection with time-varying betas and conditional variance

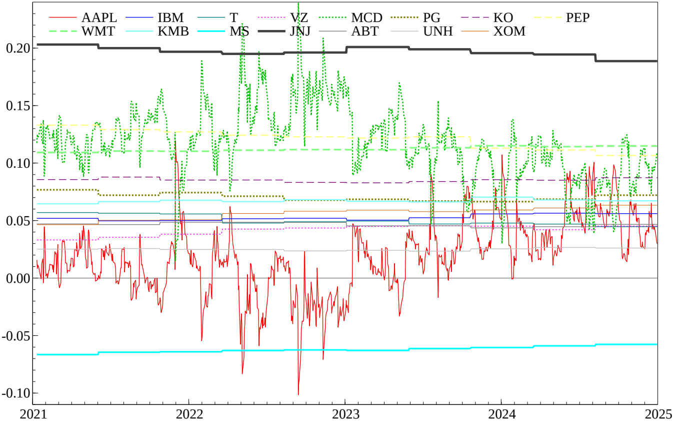
The Penalized ACB (PACB) enables variable selection with time-varying conditional betas and time-varying conditional variance:

$$y_t = \beta_{1,t} x_{1,t} + \cdots + \beta_{p,t} x_{p,t} + v_t, \quad v_t = g_t \eta_t, \quad g_t^2 \text{ is GARCH}$$

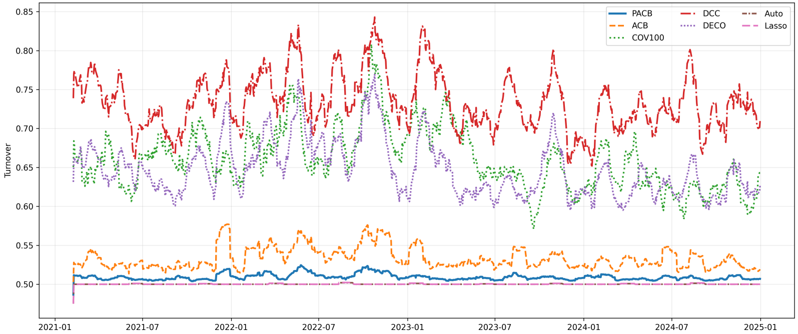
goal: for each regressor, the conditional betas is either

- time-varying, i.e., $\beta_{i,t} \neq \beta_i \forall t$;
- constant, i.e., $\beta_{i,t} = \beta_i \neq 0$;
- or null, i.e., $\beta_{i,t} = \beta_i = 0$.

Weights GMVP - PACB



Turnover $TO_t = \frac{1}{2}\sum_{i=1}^{50} |w_{i,t} - w_{i,t-1}|$.



MCS test

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Ox 10.04 (macOS_64/Parallel) (C) J.A. Doornik, 1994-2025 (oxlang.dev)
Copyright for this package: Peter R. Hansen, Asger Lunde and Sébastien Laurent
MulCom package version 3.01, object created on 22-02-2026

  ------  MODEL CONFIDENCE SET ESTIMATION  ------
Number of models:      l=7
Sample size:           n=1000
Loss function:         ident
Test Statistic:        Range
Resample by:           BlockBootResamp
Bootstrap parameters: B=10000 (resamples), d=5 (block length)

Model Name                    ident(*10^3)   MCS p-val.
PACB-KM                           0.53868    1.0000 *
-------------------------------------------------------
Lasso-KM                          0.61429    0.0000
Auto-KM                           0.62263    0.0000
ACB-KM                            0.64079    0.0086
DCC                               0.65128    0.0000
DECO                              0.91946    0.0000
COV100                            0.93566    0.0000
```

Models with the smallest sample mean squared tracking error

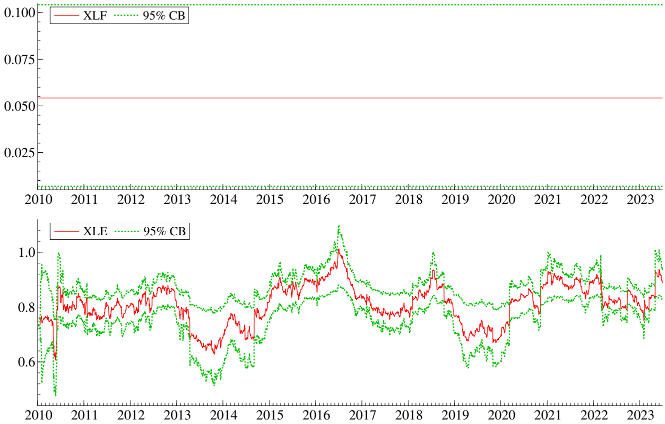
Method	# Assets	# TV weights	Var $\times 10^3$	p-value	TO_{ann}
PACB	15	2	0.53868	1.0000	1.7063
Lasso	32	0	0.61429	0.0000	0.0910
Auto	29	0	0.62263	0.0000	0.0602
ACB	50	50	0.64079	0.0086	23.6574
DCC	50	50	0.65128	0.0000	76.1992
COV100	50	50	0.91946	0.0000	86.6967
DECO	50	50	0.93566	0.0000	45.2824

Annualized TO: $TO_{ann} = 252 \times \frac{1}{999} \sum_{t=2}^{1000} TO_t$

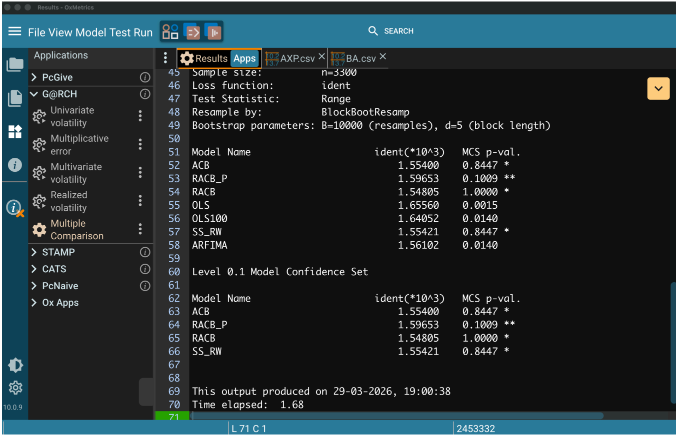
Second application: BP (British Petroleum)

We regress daily returns of BP on nine sector ETFs.

The case of BP (British Petroleum) - Oil & Gas sector



OxMetrics 10



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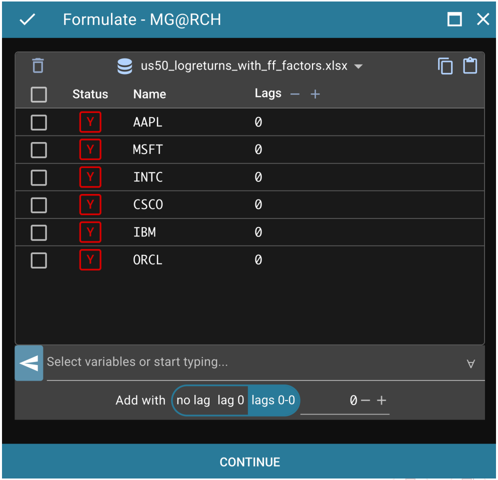
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lag 0
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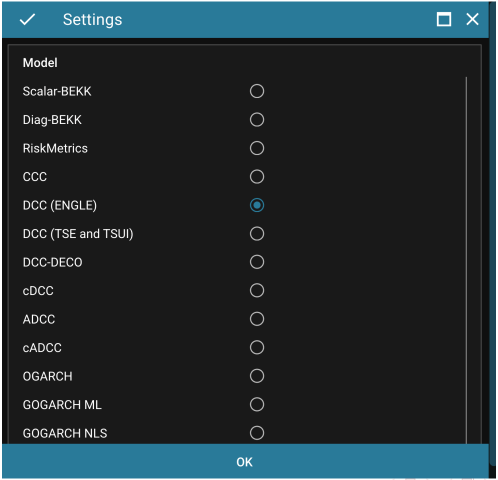
G@RCH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Diag-BEKK
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DCC-DECO
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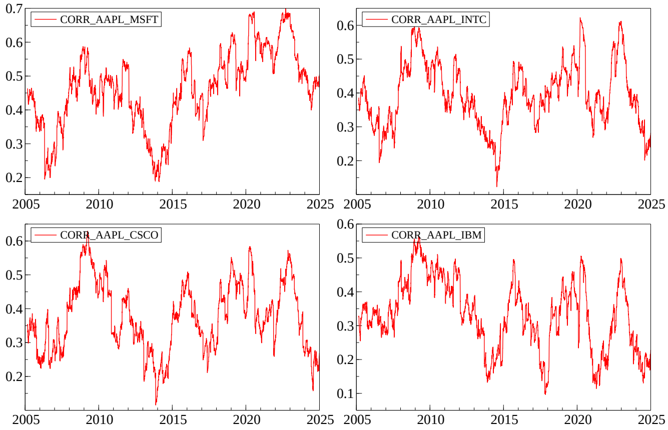
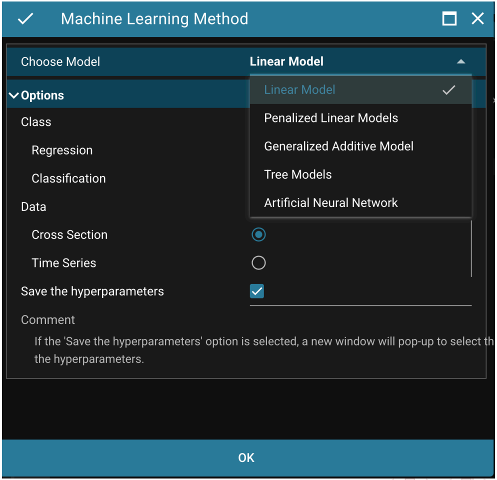
ADCC
cADCC
OGARCH
GOGARCH ML
GOGARCH NLS
OK

Correlation



OxML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Linear Model
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Linear Model
Penalized Linear Models
Generalized Additive Model
Tree Models
Artificial Neural Network
Class
Regression
Classification
Data
Cross Section
Time Series
Save the hyperparameters
Comment
If the 'Save the hyperparameters' option is selected, a new window will pop-up to select th
the hyperparameters.
OK